

C -theorem for two dimensional chiral theories

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We discuss an extension of the C -theorem to chiral theories. We show that two monotonically decreasing C -functions can be introduced. However, their difference is a constant of the renormalization group flow. This constant reproduces the 't Hooft anomaly matching conditions.

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Zamolodchikov's C -theorem [1] is an important result obtained in the studies of two dimensional quantum field theories (2D QFTs). It states that for unitary, renormalizable, local and Poincaré invariant 2D QFTs there exists a function of the couplings, $C(g^i)$, which is decreasing along the renormalization group (RG) trajectories leading to the infrared. Moreover, it states that this C -function is stationary only at the fixed points, where it coincides with the central charge of the corresponding conformal field theories. The C -theorem makes precise the intuitive idea that the detailed information on the short distance degrees of freedom is lost under the RG flow generated by the beta functions of the theory, and it shows that the RG flow is an irreversible process.

There have been several attempts directed to generalize the C -theorem to higher dimensions [2][3][4][5][6][7], but a C -theorem in 4D is still missing. Nevertheless, several hints that such a theorem would exist have been obtained working in perturbation theory [2][6], and checking explicit non-perturbative examples from supersymmetric field theories [8]. It may help to analyze the 2D C -theorem from different perspectives, with the idea of learning something which may be useful also in higher dimensions.

In this paper we consider explicitly the generic case of 2D QFTs which are allowed to break parity. We will see that for chiral theories a second C -function can be introduced. However, we will show that the difference between the two C -functions is constant along the trajectories of the renormalization group. This difference measures the amount of chiral matter that is forbidden to become massive and decouple at low energies. Effectively, this result reproduces the 't Hooft anomaly matching conditions [9], here applied to gravitational anomalies [10]. We give two different proofs of the above statements. The first proof is obtained essentially by following Zamolodchikov's footsteps. The second one is obtained by analyzing the spectral representation of the two-point function of the stress tensor, as described in the nice paper of Cappelli et al. [11]. Our interest in this second proof is to study how the extra structures related to chiral theories are encoded into the spectral representation.

We start with the first proof. Let's consider a general QFT which is unitary, renormalizable, local and Poincaré invariant. Locality and Poincaré invariance guarantee the existence of a conserved symmetric stress tensor $T_{\mu\nu}$:

$$\partial^\mu T_{\mu\nu} = 0, \quad T_{\mu\nu} = T_{\nu\mu}. \quad (1)$$

We use complex coordinates¹ and denote the independent components of the stress tensor by $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and $\Theta = T_{z\bar{z}}$. Then, the trace of the stress tensor is proportional to Θ , and the conservation laws read as follows

$$\bar{\partial}T + \partial\Theta = 0, \quad \partial\bar{T} + \bar{\partial}\Theta = 0. \quad (2)$$

Poincaré invariance fixes the general form of the two point function of the stress tensor

$$\begin{aligned} \langle T(z, \bar{z})T(0, 0) \rangle &= \frac{F(z\bar{z})}{z^4}, & \langle T(z, \bar{z})\Theta(0, 0) \rangle &= \frac{G(z\bar{z})}{z^3\bar{z}}, \\ \langle \Theta(z, \bar{z})\Theta(0, 0) \rangle &= \frac{H(z\bar{z})}{z^2\bar{z}^2}, & \langle T(z, \bar{z})\bar{T}(0, 0) \rangle &= \frac{E(z\bar{z})}{z^2\bar{z}^2}, \\ \langle \bar{T}(z, \bar{z})\bar{T}(0, 0) \rangle &= \frac{\bar{F}(z\bar{z})}{\bar{z}^4}, & \langle \bar{T}(z, \bar{z})\Theta(0, 0) \rangle &= \frac{\bar{G}(z\bar{z})}{\bar{z}^3z}, \end{aligned} \quad (3)$$

where $F, G, H, E, \bar{F}, \bar{G}$ are undetermined functions of the scalar quantity $z\bar{z}$. Chiral theories are not invariant under the exchange $z \leftrightarrow \bar{z}$. Therefore the functions F and \bar{F} , as well as G and \bar{G} , are not related to each other. Unitarity implies that the functions F, \bar{F} and H are positive, since they are related to the two-point function of one and the same operator. They can vanish only when the corresponding operator vanishes. At the fixed point, when the correlation length become infinite, we have a conformal field theory, and the trace of the stress tensor vanishes, $\Theta = 0$. Then also the functions G, \bar{G}, H and E vanish, while $F = \frac{c}{2}$ and $\bar{F} = \frac{\bar{c}}{2}$, where c and \bar{c} denote the central charges appearing in the two copies of the Virasoro algebras generated by T and \bar{T} , respectively. The fact that c and \bar{c} can have a different value in the same theory is exemplified by the case a free Weyl fermion which has $c = 1$ and $\bar{c} = 0$ (or vice-versa for the opposite chirality). This is a theory that exhibits gravitational anomalies when put on a curved space. As a consequence, it cannot be modular invariant. Imposing the conservation laws, eq. (2), on the two-point functions, eq. (3), and disregarding possible contact terms which are not important for our considerations, we obtain the following relations

$$\begin{aligned} \dot{F} + \dot{G} &= 3G, & \dot{G} + \dot{H} &= G + 2H, \\ \dot{\bar{F}} + \dot{\bar{G}} &= 3\bar{G}, & \dot{\bar{G}} + \dot{H} &= \bar{G} + 2H, \end{aligned} \quad (4)$$

¹ We work in the euclidean version of the theory by performing a Wick rotation. Nevertheless, we use a language appropriate to the minkowskian theory.

where we have defined $\dot{F} = r^2 \frac{\partial F}{\partial r^2}$ and $r^2 = z\bar{z}$. It is immediate to verify that the following combinations

$$C = 2F - 4G - 6H, \quad \bar{C} = 2\bar{F} - 4\bar{G} - 6H \quad (5)$$

satisfy

$$\dot{C} = -12H \leq 0, \quad \dot{\bar{C}} = -12H \leq 0, \quad (6)$$

where we have used the fact that H is a non-negative quantity. For a conformal field theory one has $H = 0$, and the two C -functions are stationary, taking the values $C = c$ and $\bar{C} = \bar{c}$. Defining the linear combinations

$$C_{\pm} = \frac{1}{2}(C \pm \bar{C}), \quad (7)$$

we see that

$$\dot{C}_+ = -12H \leq 0, \quad \dot{C}_- = 0. \quad (8)$$

To complete the proof of the C -theorem, we first note that the C -functions must have a form of the type

$$C = C(r^2 \Lambda^2, g^i) \quad (9)$$

where Λ denotes a mass scale parametrizing the renormalization prescription (e.g. Λ could be the renormalization point, or the mass parameter which is typically used in the minimal subtraction scheme of dimensional regularization)². In fact, the C -functions previously introduced must be dimensionless, because the stress tensor is conserved and cannot develop anomalous dimensions. In addition, it must satisfy a type of RG equation describing the arbitrariness of Λ , since the theory is assumed to be renormalizable. These two properties are expressed by the equations

$$\begin{aligned} \left(\Lambda \frac{\partial}{\partial \Lambda} - 2r^2 \frac{\partial}{\partial r^2} \right) C &= 0, \\ \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \frac{\partial}{\partial g^i} \right) C &= 0, \end{aligned} \quad (10)$$

where $\beta^i = \beta^i(g^j)$ are the beta function of the theory, guaranteed to exist as functions of the coupling constants by the renormalizability assumption. From these equations we deduce the RG equation

$$\left(2r^2 \frac{\partial}{\partial r^2} + \beta^i \frac{\partial}{\partial g^i} \right) C = 0. \quad (11)$$

² We take all coupling constants of the theory to be dimensionless by inserting appropriate powers of Λ .

Finally, we introduce a parameter t which is increasing towards the infrared direction of the RG trajectory, and integrate the β -function as

$$\frac{d}{dt}g^i = -\beta^i. \quad (12)$$

We can then compute

$$\frac{d}{dt}C_+ = -\beta^i \frac{\partial}{\partial g^i} C_+ = 2r^2 \frac{\partial}{\partial r^2} C_+ = -24H \leq 0, \quad (13)$$

where we have used eqs. (12), (11) and (6). Similarly

$$\frac{d}{dt}C_- = 0. \quad (14)$$

Thus, we see that along the RG trajectory there exist a monotonically decreasing function C_+ and a constant function C_- . Alternatively, we could say that there are two monotonically decreasing functions, C and \bar{C} , whose difference is constant along the RG flow. C and \bar{C} reduce at the fixed point to the central charges of the corresponding conformal field theory. As in ref. [1] we could set $r = 1$ in the functions C, \bar{C}, C_+ and C_- (however this is not necessary, since any fixed non-zero value of r would work).

Now for the second proof. We consider a spectral representation of the two-point function of the stress-tensor, and following ref. [11] we obtain

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = \frac{\pi}{3} \int_0^\infty d\mu \, c_i(\mu) \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{\Pi_{\mu\nu\rho\sigma}^i(p)}{p^2 + \mu^2}, \quad (15)$$

where $\Pi_{\mu\nu\rho\sigma}^i(p)$ are Lorentz covariant tensors made out of the vector p^μ and the various Lorentz invariant tensors, and $c_i(\mu)$ are the corresponding spectral functions. The tensors $\Pi_{\mu\nu\rho\sigma}^i$ must respect the symmetries of $T_{\mu\nu}$, and reproduce the conservation equation. The symmetries of $\Pi_{\mu\nu\rho\sigma}^i$ are:

1. $\Pi_{\mu\nu\rho\sigma}^i = \Pi_{\nu\mu\rho\sigma}^i, \Pi_{\mu\nu\rho\sigma}^i = \Pi_{\mu\nu\sigma\rho}^i$, which follow directly from the symmetry of $T_{\mu\nu}$.
2. $\Pi_{\mu\nu\rho\sigma}^i = \Pi_{\rho\sigma\mu\nu}^i$, which can be deduced using causality (namely, in the spectral representation one can use the fact that $[T_{\mu\nu}(x), T_{\rho\sigma}(0)] = 0$ for x spacelike. A short-cut is to note that causality also allows to derive the CPT theorem, which straightforwardly implies the above symmetry).

Finally $\Pi_{\mu\nu\rho\sigma}^i$ must reproduce conservation of the stress tensor. There are two ways to achieve this condition while respecting the above symmetries. The first is to require

that $p^\mu \Pi_{\mu\nu\rho\sigma}^i = 0$. The general solution, obtained by using the vector p^μ and the Lorentz invariant tensors $g_{\mu\nu}$ and $\epsilon_{\mu\nu}$, is

$$\Pi_{\mu\nu\rho\sigma}^1 = (p_\mu p_\nu - g_{\mu\nu} p^2)(p_\rho p_\sigma - g_{\rho\sigma} p^2). \quad (16)$$

We note that after defining $\tilde{p}_\mu = \epsilon_{\mu\nu} p^\nu$ the above solution can be written as

$$\Pi_{\mu\nu\rho\sigma}^1 = \tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho \tilde{p}_\sigma. \quad (17)$$

The use of $\epsilon_{\mu\nu}$ has added no new solutions above those given in [11]. The corresponding spectral function $c_1(\mu)$, which has mass dimension $d = -1$, can be parametrized as

$$c_1(\mu) = c_1 \delta(\mu) + \tilde{c}_1(\mu, \Lambda), \quad (18)$$

where Λ is a mass scale of the theory (a term proportional to μ^{-1} is not allowed since it would not make the spectral representation convergent in the infrared). As described in ref. [11], the delta function term represents the degrees of freedom at arbitrarily large distances, while $\tilde{c}_1(\mu, \Lambda)$ is due to the density of degrees of freedom at distances μ^{-1} .

A second way to achieve conservation works only at $\mu = 0$. Namely, we require that $p^\mu \Pi_{\mu\nu\rho\sigma} = p^2 \Pi_{\nu\rho\sigma}$, with an undetermined tensor $\Pi_{\nu\rho\sigma}$. In fact, the p^2 factor then cancels the pole at $\mu = 0$ in eq. (15), and we get conservation of the stress tensor up to contact terms (which are generically present in any case). The independent solutions to this equation, which do not give rise to purely contact terms, are

$$\begin{aligned} \Pi_{\mu\nu\rho\sigma}^2 &= p_\mu^+ p_\nu^+ p_\rho^+ p_\sigma^+, \\ \Pi_{\mu\nu\rho\sigma}^3 &= p_\mu^- p_\nu^- p_\rho^- p_\sigma^-, \end{aligned} \quad (19)$$

where $p_\mu^\pm = \frac{1}{2}(p_\mu \mp i\tilde{p}_\mu)$ are the light-cone projections of the momentum (note that these solutions are real in the minkowskian theory). As described, the corresponding spectral functions must be localized at $\mu = 0$, i.e. $c_k(\mu) = c_k \delta(\mu)$ for $k = 2, 3$. However, the minimum between c_2 and c_3 , $c_{min} = \min\{c_2, c_3\}$, does not define an universal quantity. It can be absorbed into the constant c_1 of eq. (18) by adding local terms to the effective action. This corresponds to a redefinition of the renormalization scheme. This statement is easily checked by writing out explicitly the components of the various tensors in a light-cone coordinate basis. On the pole $p^2 = 4p_z p_{\bar{z}}$ only the components $\Pi_{zzzz}^2 = p_z^4$ and $\Pi_{\bar{z}\bar{z}\bar{z}\bar{z}}^3 = p_{\bar{z}}^4$ give rise to non-local terms. When the spectral coefficients of these two quantities are the same, one can reconstruct the tensor (16) by adding local terms to the effective action.

This proves that it is consistent to set $c_{min} = 0$ by choosing a suitable renormalization scheme. On the other hand, the quantity $c_2 - c_3$ has an invariant meaning (i.e. independent on the renormalization scheme). One can immediately check that $C_- = \frac{1}{2}(c_2 - c_3)$. This quantity measures the amount of chiral matter that is forbidden to become massive and it is an invariant of the RG flow. In fact, the corresponding pole present in the spectral representation (15) must necessarily be at $\mu = 0$, otherwise the conservation of $T_{\mu\nu}$ is not achieved. This massless pole is related to the gravitational anomalies that would arise if the theory is put on a curved background. Thus we recognize that the RG invariance of C_- is consistent with (and required by) the 't Hooft anomaly matching conditions, according to which the infrared limit of a theory must arrange itself in such a way to recreate the massless singularities responsible for the chiral anomalies, the latter being typically computed in the ultraviolet regime. Finally, a standard flowing Zamolodchikov's C -function can be obtained by smearing $c_1(\mu)$ against a density function $f(\mu)$

$$C_+ = \int d\mu f(\mu) c_1(\mu) \quad (20)$$

satisfying the properties $f(\mu) > 0$, $f(0) = 1$, $f(\mu)$ decreasing exponentially as $\mu \rightarrow \infty$, and $\mu \frac{d}{d\mu} f \leq 0$. In fact, one can compute

$$\frac{d}{dt} C_+ = -\beta^i \frac{\partial}{\partial g^i} C_+ = \Lambda \frac{\partial}{\partial \Lambda} C_+ = \int d\mu \tilde{c}_1(\mu, \Lambda) \mu \frac{d}{d\mu} f(\mu) \leq 0, \quad (21)$$

where we have used a Callan-Symanzik RG equation, employed dimensional analysis for $\tilde{c}_1(\mu, \Lambda)$, integrated by parts and used unitarity that constrains the spectral functions to be positive. The function $f(\mu)$ can be chosen so that C_+ coincides with the function defined in eq. (7), (see ref. [11]).

In summary, we have analyzed an extension of the C -theorem to chiral theories. We have seen that two C -functions can be introduced, C and \bar{C} , which are monotonically decreasing along the RG flow. However, their difference is constant and it is related to the amount of chiral matter that cannot decouple at low energy by becoming massive. This result easily explains why 't Hooft anomaly matching conditions work in this two dimensional case. In fact, our result can be considered to be an alternative, easier proof of such a matching. We have also employed a spectral representation of the two point function of the stress tensors to unearth how the extra structures related to chiral theories are encoded into the spectral functions.

In 4D gravitational anomalies are absent as a consequence of CPT invariance, and so it would seem that considering chiral structures in 4D would not be of much help. However, in our analysis we have shown how some, apparently unrelated, things fit nicely together. Such insight may help, after all, also in understanding the 4D problem.

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